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On some inequalities for Walsh—Fourier series

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Let $f(x)$ be a distribution on $[0, 1]$ and its Walsh—Fourier series be $\sum_{n=0}^{\infty} \hat{f}(n) W_n(x)$, $\hat{f}(n)=(f, W_n)$. The Littlewood—Paley function $g(f)(x)$ is defined by

$$\{\sum_{n=1}^{\infty} n(\sigma_{n+1} f(x)-\sigma_n f(x))^2 a_n\}^{1/2},$$

where $\sigma_n f(x)=\sum_{k=0}^{n-1}\left(1-\frac{k}{n}\right) \hat{f}(k) W_k(x)$ and $\{a_n\}$ is a sequence of positive constants satisfying some conditions. The Marcinkiewicz multiplier operator M is given by

$$Mf(x) \sim \sum_{k=0}^{\infty} \lambda(k) \hat{f}(k) W_k(x),$$

where $\{\lambda(k)\}$ is bounded and varies boundedly over each dyadic block.

We shall show some inequalities for $g(f)(x)$ and $Mf(x)$ using Zygmund's inequalities.

Let $r_0(x)=\operatorname{sgn} \sin 2\pi x$, and $r_n(x)=r_0(2^n x)$. The Walsh—Paley functions are defined as follows:

$$w_0(x)=1; \quad w_n(x)=r_{n_1}(x) \ldots r_{n_k}(x), \quad \text{if} \quad n=2^{n_1}+\ldots+2^{n_k}, \quad n_1>n_2>\ldots>n_k \geqq 0.$$

The collection $\{w_n(x);\ n=0, 1, 2, \ldots\}$ forms a complete orthonormal system for L^2 over the unit interval $0 \leqq x \leqq 1$.

Let S be the collection of Walsh polynomials, and S' be the space of distributions on $0 \leqq x \leqq 1$. If $f \in S'$, the Fourier coefficients $\{\hat{f}(n)\}_{n=0}^{\infty}$ are given by $\hat{f}(n)=(f, w_n)$, where (f, w_n) denotes the action of $f \in S'$ on $w_n \in S$. The Fourier series is

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given by $f(x) \sim \sum_{n=0}^{\infty} \hat{f}(n) w_n(x)$. Write

$$S_n f(x) = \sum_{k=0}^{n-1} \hat{f}(k) w_k(x), \quad \sigma_n f(x) = \sum_{k=0}^{n-1} \left(1-\frac{k}{n}\right) \hat{f}(k) w_k(x),$$

$$d_n f(x) = S_{2^n} f(x) - S_{2^{n-1}} f(x), \quad d_0 f(x) = \hat{f}(0), \quad \text{and} \quad Sf(x) = \left(\sum_{n=0}^{\infty} |d_n f(x)|^2\right)^{1/2}.$$

For $0<p<\infty$, let H^p be the space of $f \in S'$ whose $Sf(x) \in L^p$ with the H^p-norm $\|Sf\|_{L^p}$.

Let $f \in S'$, and the Littlewood-Paley function be

$$g(f)(x) = \left\{\sum_{n=1}^{\infty} n\left(\sigma_{n+1} f(x) - \sigma_n f(x)\right)^2 a_n\right\}^{1/2},$$

where $\{a_n\}$ is a sequence of positive constants satisfying $c \cdot n \leq \sum_{k=1}^{n} a_k \leq C \cdot n$ and $\sum_{k=2^n}^{2^{n+1}-1} (a_k)^{-1} \leq C \cdot 2^n$ for all n and some positive constans c and C. As special cases, if we take $a_n = 1$ for all n, then

$$g(f)(x) = \left\{\sum_{n=1}^{\infty} \left(\sigma_{n+1} f(x) - S_{n+1} f(x)\right)^2 / n\right\}^{1/2}$$

and if we take $a_n = n$ $(n = 2^k - 1)$, 0 (otherwise), then

$$g(f)(x) = \left\{\sum_{n=0}^{\infty} \left(S_{2^n} f(x) - \sigma_{2^n} f(x)\right)^2\right\}^{1/2}.$$

By c, C and C_p we always denote a positive constant that may be different on various occasions. We can prove the following theorem.

Theorem 1. (1) *If* $f \in H^1$ *and* $\lambda > 0$, *then*

$$m(\{x \in [0, 1]: g(f)(x) > \lambda\}) \leq C \|f\|_{H^1} / \lambda$$

(2) *If* $f \in H^1$, *then* $\|g(f)\|_{L^p} \leq C_p \|f\|_{H^1}$ $(0 < p < 1)$.

(3) *If* $Sf \in L^1 \log^+ L^1$, *then* $\|g(f)\|_{L^1} \leq C \|Sf\|_{L^1 \log^+ L^1} + C$.

(4) *If* $\hat{f}(0) = 0$ *and* $g(f) \in L^1$, *then* $\|f\|_{H^p} \leq C \|g(f)\|_{L^1}$ $(0 < p < 1)$.

To prove Theorem 1 we need the following. Let $F(x) = \{f_1(x), f_2(x), \ldots\}$ be a l^2-valued function of $x \in [0, 1]$ and measurable in the Bochner sense. Write $|F(x)| = \left(\sum_{k=1}^{\infty} |f_k(x)|^2\right)^{1/2}$. Similarly let $N = \{n(k)\}$ be an arbitrary sequence of integer-valued function of k, and write $S_N(F)(x) = \{S_{n(1)}(f_1)(x), S_{n(2)}(f_2)(x), \ldots\}$. Using this notation, analogue of Zygmund's inequalities can then be stated as

Lemma. (1) *For any* $\lambda>0$,

$$m(\{x\in[0,1]:\ |S_N(F)(x)|>\lambda\})\leq\int|F(x)|\,dx/\lambda,$$

(2) $$\int|S_N(F)(x)|^p\,dx\leq C_p\Big(\int|F(x)|\,dx\Big)^p,\quad(0<p<1),$$

(3) $$\int|S_N(F)(x)|\,dx\leq C\int|F(x)|\log^+|F(x)|\,dx+c.$$

(1) is due to G. Sunouchi [3]. (2) is due to W. R. Wade [4]. Proof of (3) is the same as (2), using the inclusion $L^1\log^+L^1\subset H^1\subset L^1$, Khinchin's inequality and Paley's decomposition.

Proof of Theorem 1. By an identity due to E. M. Stein [2, p. 114],

$$\sigma_{n+1}f(x)-\sigma_nf(x)=$$

$$=\frac{1}{n(n+1)}\Big[\sum_{j=0}^{[\log_2 n]}(2^j-1)\,d_jf(x)+nS_{n+1}(d_{1+[\log_2 n]}f)(x)-\sum_{k=1}^{n}e_kS_k(d_{j(k)}f)(x)\Big],$$

where $j(k)$ is an appropriate integer-valued function of k and e_k is 0 or 1. Then

$$g(f)(x)\leq\Big(\sum_{n=1}^{\infty}a_nn^{-2}(n+1)^{-1}\Big|\sum_{j=0}^{[\log_2 n]}(2^j-1)\,d_jf(x)\Big|^2\Big)^{1/2}+$$

$$+\Big(\sum_{n=1}^{\infty}a_n(n+1)^{-1}|S_{n+1}(d_{1+[\log_2 n]}f)(x)|^2\Big)^{1/2}+$$

$$+\Big(\sum_{n=1}^{\infty}a_nn^{-2}(n+1)^{-1}\Big|\sum_{k=1}^{n}e_kS_k(d_{j(k)}f)(x)\Big|^2\Big)^{1/2}=A(x)+B(x)+C(x).$$

For $0<p<2$, by Khinchin's inequality,

$$\|A\|_{L^p}^p\leq C\Big\|\int\Big|\sum_{n=1}^{\infty}(a_n\cdot n^{-3})^{1/2}\Big(\sum_{j=0}^{[\log_2 n]}(2^j-1)\,d_jf\Big)r_n(t)\Big|^p\,dt\Big\|_1.$$

From $\|f\|_{L^p}\leq\|f\|_{H^p}$, Hölder's inequality and the condition on $\{a_n\}$,

$$\|A\|_{L^p}^p\leq C\int dt\int\Big\{\sum_{j=0}^{\infty}2^{2j}\Big(\sum_{n=2^j}^{\infty}(a_n\cdot n^{-3})^{1/2}r_n(t)\Big)^2(d_jf(x))^2\Big\}^{p/2}dx\leq$$

$$\leq C\int dx\Big\{\int\sum_{j=0}^{\infty}2^{2j}(d_jf(x))^2\Big(\sum_{n=2^j}^{\infty}(a_n\cdot n^{-3})^{1/2}r_n(t)\Big)^2dt\Big\}^{p/2}\leq$$

$$=C\int\Big\{\sum_{j=0}^{\infty}2^{2j}(d_jf(x))^2\sum_{n=2^j}^{\infty}(a_n\cdot n^{-3})\Big\}^{p/2}dx=C\|Sf\|_{L^p}^p.$$

Thus, $\|A\|_{L^p}\leq C\|f\|_{H^p}$. By the condition on $\{a_n\}$, we have

$$B(x)\leq C\Big(\sum_{n=1}^{\infty}|S_{n+1}(d_{1+[\log_2 n]}f)|^2\Big)^{1/2},$$

and

$$C(x) \leqq \Big(\sum_{k=1}^{\infty} |S_k(d_{j(k)}f)(x)|^2 \sum_{n=k}^{\infty} a_n \cdot n^{-2}\Big)^{1/2} \leqq C\Big(\sum_{k=1}^{\infty} |S_k(d_{j(k)}f)(x)|^2\Big)^{1/2}.$$

We can now prove part 1 of Theorem 1, by using the above estimates for A, B and C, and Lemma (1). For if $f \in H^1$, then we have

$$m(\{x \in [0, 1]:\ g(f)(x) > \lambda\}) \leqq$$

$$\leqq m(\{x:\ A(x) > \lambda/3\}) + m(\{x:\ B(x) > \lambda/3\}) + m(\{x:\ C(x) > \lambda/3\}) \leqq$$

$$\leqq C\|f\|_{H^1}/\lambda + m\Big(\Big\{x:\ \Big(\sum_{n=1}^{\infty} |S_{n+1}(d_{1+[\log_2 n]}f)(x)|^2\Big)^{1/2} > C\cdot\lambda/3\Big\}\Big) +$$

$$+ m\Big(\Big\{x:\ \Big(\sum_{n=1}^{\infty} |S_n(d_{j(n)}f)(x)|^2\Big)^{1/2} > \lambda/3\Big\}\Big) \leqq C\|f\|_{H^1}/\lambda.$$

Similarly we can easily prove part 2 and 3 of Theorem 1. To prove part 4 of Theorem 1, write

$$Sf(x) = \Big(\sum_{n=0}^{\infty} |S_{2^{n+1}}f(x) - S_{2^n}f(x)|^2\Big)^{1/2} \leqq$$

$$\leqq 2\Big(\sum_{n=0}^{\infty} |S_{2^n}f(x) - \sigma_{2^n}f(x)|^2\Big)^{1/2} + \Big(\sum_{n=0}^{\infty} |\sigma_{2^{n+1}}f(x) - \sigma_{2^n}f(x)|^2\Big)^{1/2}.$$

By the condition on $\{a_n\}$ and Schwartz's inequality,

$$|\sigma_{2^{n+1}}f(x) - \sigma_{2^n}f(x)| \leqq \sum_{k=2^n}^{2^{n+1}-1} |\sigma_{k+1}f(x) - \sigma_k f(x)| \leqq$$

$$\leqq \Big\{\sum_{k=2^n}^{2^{n+1}-1} k\big(\sigma_{k+1}f(x) - \sigma_k f(x)\big)^2 a_k\Big\}^{1/2}.$$

Hence

$$\Big(\sum_{n=1}^{\infty} |\sigma_{2^{n+1}}f(x) - \sigma_{2^n}f(x)|^2\Big)^{1/2} \leqq g(f)(x).$$

On the other hand, it is evident to see

$$\Big(\sum_{n=0}^{\infty} |S_{2^n}f(x) - \sigma_{2^n}f(x)|^2\Big)^{1/2} = \Big(\sum_{n=0}^{\infty} 2^{-2n}\Big|\sum_{j=0}^{2^n-1} j\hat{f}(j)\,w_j(x)\Big|^2\Big)^{1/2} \leqq$$

$$\leqq C\Big(\sum_{n=0}^{\infty} \sum_{k=2^n}^{2^{n+1}-1} a_k k^{-1}(k+1)^{-2}\Big|S_{2^n}\Big(\sum_{j=0}^{k} j\hat{f}(j)\,w_j(x)\Big)\Big|^2\Big)^{1/2}.$$

Thus, by Lemma (2), for $0<p<1$; $\|Sf\|_{L^p}^p \leqq C\|g(f)\|_{L^1}^p$. This completes the proof of Theorem 1.

Remark. It is easily verified that $f \to g(f)$ is strong type (2.2). Therefore, by Theorem 1 (1) and interpolation argument, we have $\|g(f)\|_{L^p} \leqq C_p \|f\|_{L^p}$ $(1<p<\infty)$. On the other hand $f \to g(f)$ is not weak type (1, 1) for L^1. See S. IGARI [1].

Next we study Marcinkiewicz multiplier theorem. Let $\{\lambda(k)\}$ be a sequence of constants such that

$$\sup_k |\lambda(k)| \leqq C, \quad \sup_j \sum_{k=2^j+1}^{2^{j+1}-1} k|\Delta\lambda(k)|^2 \leqq C, \quad \text{where} \quad \Delta\lambda(k) = \lambda(k-1)-\lambda(k),$$

and consider the linear transformation M, defined by

$$Mf(x) \sim \sum \lambda(k) \hat{f}(k) w_k(x) \quad \text{for} \quad f(x) \sim \sum \hat{f}(k) w_k(x).$$

Theorem 2. *Under the assumption made above,*

(1) $$m(\{x \in [0, 1]: S(Mf)(x) > \lambda\}) \leqq C\|f\|_{H^1}/\lambda,$$

(2) $$\|Mf\|_{H^p} \leqq C_p \|f\|_{H^1} \quad (0<p<1),$$

(3) $$\|Mf\|_{H^1} \leqq C\|Sf\|_{L^1 \log^+ L^1} + c.$$

Proof. By summation by part,

$$d_j(Mf)(x) = \sum_{k=2^{j-1}}^{2^j-1} \lambda(k) \hat{f}(k) w_k(x) =$$

$$= \sum_{k=2^{j-1}+1}^{2^j-1} \Delta\lambda(k) S_k(d_j f)(x) + \lambda(2^j-1)\, d_j f(x).$$

Then, by Schwartz's inequality and assumption of $\{\lambda(k)\}$,

$$S(Mf)(x) = (\sum |d_j(Mf)(x)|^2)^{1/2} \leqq$$

$$\leqq \{\sum_j (\sum_{k=2^{j-1}+1}^{2^j-1} k|\Delta\lambda(k)|^2)(\sum_{k=2^{j-1}+1}^{2^j-1} k^{-1}|S_k(d_j f)(x)|^2\}^{1/2} + C(\sum_j |d_j f(x)|^2)^{1/2} \leqq$$

$$\leqq C(\sum_{j,k} |S_k(d_j f)(x)|^2)^{1/2} + CSf(x).$$

Thus Theorem 2 is proved by the Lemma.

Remark. The same argument works for double Walsh—Fourier series and Vilenkin—Fourier series.

References

[1] S. Igari, An extension of the interpolation theorem of Marcinkiewicz, *Tôhoku Math. J.*, **15** (1963), 343—358.
[2] E. M. Stein, Topics in harmonic analysis related to the Littlewood—Paley theory, *Ann. of Math. Stud.*, 63 (Princeton, 1970).
[3] G. Sunouchi, Strong summability of Walsh Fourier series, *Tôhoku Math. J.*, **16** (1964), 228—237.
[4] W. R. Wade, L^r inequalities for Walsh series, $0<r<1$, *Acta Sci. Math.*, **46** (1983), 233—241.

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